Ex 13

We simplify the formula using the identity $x \vee \neg x \wedge y = x \vee y$

$$\begin{aligned}
(a \vee b) \wedge (\neg b \vee c) \Rightarrow a \vee c &= \neg((a \vee b) \wedge (\neg b \vee c)) \vee a \vee c \\
&= \neg(a \vee b) \vee \neg(\neg b \vee c) \vee a \vee c \\
&= (\neg a \wedge \neg b) \vee (\neg\neg b \wedge \neg c) \vee a \vee c \\
&= a \vee (\neg a \wedge \neg b) \vee c \vee (\neg c \wedge b) \\
&= a \vee \neg b \vee c \vee b \\
&= a \vee c \vee \neg b \vee b \\
&= a \vee c \vee 1 \\
&= 1
\end{aligned}$$

Ex 16

Reminder: given an operator $*$, we say that $*$ is:

- commutative if for any x, y, $x * y = y * x$.
- associative if for any x, y, z, $(x * y) * z = x * (y * z)$.
- idempotent if for any x, $x * x = x$.
- transitive if for any x, y, z, $x * y$ and $y * z$ implies $x * z$.

Idempotence

x	$x \Rightarrow x$	$x \Leftrightarrow x$
0	1	1
1	1	1

According to the truth table, $\Rightarrow$ and $\Leftrightarrow$ are not idempotent.

Communtativity

x	y	$x \Rightarrow y$	$y \Rightarrow x$	$x \Leftrightarrow y$	$y \Leftrightarrow x$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

According to the truth table, $\Rightarrow$ is not commutative, but $\Leftrightarrow$ is.

Associativity

x	y	z	$x \Rightarrow (y \Rightarrow z)$	$(x \Rightarrow y) \Rightarrow z$	$x \Leftrightarrow (y \Leftrightarrow z)$	$(x \Leftrightarrow y) \Leftrightarrow z$
0	0	0	1	0	0	0
0	0	1	1	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	0	0
1	0	0	1	1	1	1
1	0	1	1	1	0	0
1	1	0	0	0	0	0
1	1	1	1	1	1	1

According to the truth table, $\Rightarrow$ is not associative but $\Leftrightarrow$ is.

Transitivity

x	y	z	$x \Rightarrow y$	$y \Rightarrow z$	$x \Rightarrow z$	$x \Leftrightarrow y$	$y \Leftrightarrow z$	$x \Leftrightarrow z$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	0	0
0	1	0	1	0	1	0	0	1
0	1	1	1	1	1	0	1	0
1	0	0	0	1	0	0	1	0
1	0	1	0	1	1	0	0	1
1	1	0	1	0	0	1	0	0
1	1	1	1	1	1	1	1	1

According to the truth table, $\Rightarrow$ and $\Leftrightarrow$ are transitive.

Ex 23

Transforming $\neg(a \Leftrightarrow b) \vee (b \wedge c) \Rightarrow c$ in DNF.

$$\begin{aligned}
\neg(a \Leftrightarrow b) \vee (b \wedge c) \Rightarrow c &\equiv \neg((a \Rightarrow b) \wedge (b \Rightarrow a)) \vee (b \wedge c) \Rightarrow c \\
&\equiv \neg((\neg a \vee b) \wedge (\neg b \vee a)) \vee (b \wedge c) \Rightarrow c \\
&\equiv ((\neg a \vee b) \wedge (\neg b \vee a) \wedge (\neg b \vee \neg c)) \vee c \\
&\equiv (((\neg a \wedge \neg b) \vee (b \wedge a)) \wedge (\neg b \vee \neg c)) \vee c \\
&\equiv (((\neg a \wedge \neg b) \vee (b \wedge a)) \vee c) \wedge (\neg b \vee \neg c \vee c) \\
&\equiv \neg a \wedge \neg b \vee a \wedge b \vee c
\end{aligned}$$

Any model of this formula must respect one of the three following conditions:

- **$a = 1$ and $b = 1$.**
- **$a = 0$ and $b = 0$.**
- **$c = 1$.**

Transforming $(a \Rightarrow b) \wedge (b \Rightarrow \neg a) \wedge (\neg a \Rightarrow b) \wedge (b \Rightarrow a)$ in DNF.

$$\begin{aligned}(a \Rightarrow b) \wedge (b \Rightarrow \neg a) \wedge (\neg a \Rightarrow b) \wedge (b \Rightarrow a) &\equiv (\neg a \vee b) \wedge (\neg b \vee \neg a) \wedge (a \vee b) \wedge (\neg b \vee a) \\ &\equiv \neg a \wedge a \\ &\equiv 0\end{aligned}$$

This formula is a contradiction, thus it has no model.

Ex 27

Notation: a: Aha is a Tame, b: Beeby is a Tame.

"Aha says: at least one of us is a Lame" We can represent this assertion (the fact that Aha says that one of them is a Lame) by:

$$a \Leftrightarrow \neg a \vee \neg b$$

. This formula is equivalent to $a \wedge \neg b$ (the simplification of the formula is left as an exercice for the reader). Hence, we conclude that Aha is a Tame, and Beeby is a Lame.

"Aha says: at most one of us is a Lame" We can represent this assertion by

$$a \Leftrightarrow a \vee b$$

, which is equivalent to $a \vee \neg b$. So we cannot conclude.

"Aha says: both of us are in the same tribe" This assertion is equivalent to $a \Leftrightarrow (a \Leftrightarrow b)$. We can reduce this formula to its normal form, b. So we conclude that we know that Beeby is a Tame. We cannot conclude for Aha.