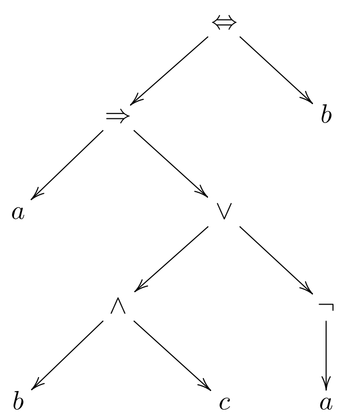
Exercise 1

Let $A = a \Rightarrow b \wedge c \vee \neg a \Leftrightarrow b$.

Corresponding strict formula: $((a \Rightarrow ((b \wedge c) \vee \neg a)) \Leftrightarrow b)$

Corresponding tree:



Truth table:

a	b	c	$b \wedge c$	$\neg a$	$(b \wedge c) \vee \neg a$	$a \Rightarrow ((b \wedge c) \vee \neg a)$	A
0	0	0	0	1	1	1	0
0	0	1	0	1	1	1	0
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	1
1	0	1	0	0	0	0	1
1	1	0	0	0	0	0	0
1	1	1	1	0	1	1	1

Simplification into DNF:

$$
\begin{aligned}
A &\equiv (a \Rightarrow (bc + \bar{a})) \Leftrightarrow b \\
&\equiv (\bar{a} + bc + \bar{a}) \Leftrightarrow b \\
&\equiv (\bar{a} + bc) \Leftrightarrow b \\
&\equiv ((\bar{a} + bc) \cdot b) + (\overline{(\bar{a} + bc)} \cdot \bar{b}) \\
&\equiv ((\bar{a} + bc) \cdot b) + ((a \cdot \overline{bc}) \cdot \bar{b}) \\
&\equiv ((\bar{a} + bc) \cdot b) + (a \cdot (\bar{b} + \bar{c}) \cdot \bar{b}) \\
&\equiv (\bar{a}b + bc) + ((a\bar{b} + a\bar{c}) \cdot \bar{b}) \\
&\equiv (\bar{a}b + bc) + ((a\bar{b}\bar{b} + a\bar{c}\bar{b})) \\
&\equiv (\bar{a}b + bc) + (a\bar{b} + a\bar{c}\bar{b}) \\
&\equiv \bar{a}b + bc + a\bar{b} + a\bar{c}\bar{b}
\end{aligned}
$$

Simplification into CNF:

$$\begin{aligned}
A &\equiv \bar{a}b + bc + a\bar{b} + a\bar{c}\bar{b} \\
&\equiv (\bar{a}+b+a+a)\cdot(\bar{a}+b+a+\bar{c})\cdot(\bar{a}+b+a+\bar{b}) \\
&\quad \cdot(\bar{a}+b+\bar{b}+a)\cdot(\bar{a}+b+\bar{b}+\bar{c})\cdot(\bar{a}+b+\bar{b}+\bar{b}) \\
&\quad \cdot(\bar{a}+c+a+a)\cdot(\bar{a}+c+a+\bar{c})\cdot(\bar{a}+c+a+\bar{b}) \\
&\quad \cdot(\bar{a}+c+\bar{b}+a)\cdot(\bar{a}+c+\bar{b}+\bar{c})\cdot{\color{red}(\bar{a}+c+\bar{b}+\bar{b})} \\
&\quad \cdot{\color{red}(b+b+a+a)}\cdot(b+b+a+\bar{c})\cdot(b+b+a+\bar{b}) \\
&\quad \cdot(b+b+\bar{b}+a)\cdot(b+b+\bar{b}+\bar{c})\cdot(b+b+\bar{b}+\bar{b}) \\
&\quad \cdot(b+c+a+a)\cdot(b+c+a+\bar{c})\cdot(b+c+a+\bar{b}) \\
&\quad \cdot(b+c+\bar{b}+a)\cdot(b+c+\bar{b}+\bar{c})\cdot(b+c+\bar{b}+\bar{b}) \\
&\equiv (c+\bar{a}+\bar{b})\cdot(a+b)
\end{aligned}$$

Exercise 2

We formalize the facts:

- e: The TD lecturer gives hard exercises;
- r: the students get better results;
- h: the students hate the TD lecturer.

We can then formalize the two hypotheses and transform them in products of clauses:

1. $e \Rightarrow rh \equiv \bar{e} + rh \equiv (\bar{e}+r)\cdot(\bar{e}+h)$
2. $h \Rightarrow e \equiv \bar{h} + e$

We also model the conclusion: $\bar{r} \Rightarrow \bar{e} + \bar{h} \equiv r + \bar{e} + \bar{h}$.

We transform the negation of the conclusion into a product of clauses as well:

$\overline{r + \bar{e} + \bar{h}} \equiv \overline{r + \bar{e}}h \equiv \bar{r}eh$

To prove that the reasoning is correct, we have to show that the conjunction of the hypotheses and of the negation of the conclusion is unsatisfiable.

Corresponding set of clauses: $\Gamma = \{\bar{e}+r, \bar{e}+h, e+h, \bar{r}, e, h\}$.

We can prove it using several methods: truth table, expression simplification (proving that it is equivalent to $\bot$), or propositional resolution. Any of these three methods was accepted.

Exercise 3

For any $n \in \mathbb{N}$, we define $\mathcal{H}(n)$: "Any strict formula of size n is also a priority formula".

Base case: Let A be a strict formula such that $|A| = 0$. Then, by definition of strict formulae, we have:

- $A = \top$ or
- $A = \bot$ or
- $A = x$ with x a variable.

In any of these 3 cases, A is also a priority formula, so $\mathcal{H}(0)$ holds. We have proved the base case.

Heredity: Let $n \in \mathbb{N}$. We suppose that $\mathcal{H}(k)$ is true for any $k \leq n$. Let A be a strict formula such that $|A| = n+1$. Since $|A| > 0$, there are two possibilities:

- $A = \neg B$, with B a strict formula. We then have $|B| = |A| - 1 = n$. By induction, B is also a priority formula. By definition of priority formulae, so is A.
- $A = (B \circ C)$, with B, C strict formulae and $\circ \in \{\vee, \wedge, \Rightarrow, \Leftrightarrow\}$. We then have $|B| + |C| = n$, with $|B| \geq 0$ and $|C| \geq 0$, so we have $|B| \leq n$ and $|C| \leq n$. By induction, B and C are priority formulae. By definition of priority formulae, $B \circ C$ is also a priority formulae, and $(B \circ C)$ as well. So A is a priority formula.

The heredity property holds. We can conclude by saying that any strict formula is also a priority formula.